Journal of Geometry and Physics 39 (2001) 45–49
JOURNAL OF GEOMETRY AND PHYSICS

Remarks on compact spacelike hypersurfaces in de Sitter space with constant higher order mean curvature

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Received 4 September 2000

Abstract

It is shown that a compact spacelike hypersurface which is contained in the chronological future (or past) of an equator of de Sitter space is a totally umbilical round sphere if one of the mean curvatures H_l does not vanish and the ratio H_k/H_l is constant for some k, l, $1 \leq l < k \leq n$. This extends the previous result in [J. Geom. Phys. 31 (1999) 195, Theorem 7]. © 2001 Elsevier Science B.V. All rights reserved.

MSC: Primary 53C42; Secondary 53B30; 53C50

*Subj. Class.:*General relativity; Differential geometry

Keywords: de Sitter space; Spacelike hypersurfaces; Higher order mean curvatures; Minkowski formulas

1. Introduction and statement of the main results

Let $\mathbf{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space, i.e., the real vector space $\mathbf{R}^{n+2}$ endowed with the Lorentzian metric tensor $\langle , \rangle$ given by

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

and let $\mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be the $(n+1)$-dimensional unitary de Sitter space, i.e.,

$$\mathbf{S}_1^{n+1} = \{x \in \mathbf{L}^{n+2} : \langle x, x \rangle = 1\}.$$

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PII: S0393-0440(00)00073-5

It is well known that, for $n \geq 2$ the de Sitter space $\mathbf{S}_1^{n+1}$ is the standard simply connected Lorentzian space form of positive constant sectional curvature. A smooth immersion $\psi : M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ of an n-dimensional connected manifold M^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on M^n, which, as usual, is also denoted by $\langle , \rangle$.

The interest for the study of spacelike hypersurfaces in de Sitter space is motivated by the fact that such hypersurfaces exhibit nice Bernstein-type properties. In 1977, Goddard [5] conjectured that the only complete spacelike hypersurfaces with constant mean curvature in $\mathbf{S}_1^{n+1}$ should be the totally umbilical ones. This conjecture, which turned out to be false in its original statement, motivated the work of an important number of authors who considered the problem of characterizing the totally umbilical spacelike hypersurfaces of de Sitter space in terms of some appropriate geometric assumptions. In particular, Akutagawa [1] showed that Goddard's conjecture is true if the constant mean curvature H of the hypersurface satisfies $0 \leq H^2 \leq 1$ when $n = 2$ and $0 \leq H^2 < 4(n-1)/n^2$ in the general case $n \geq 3$. As an application of it, Akutagawa also proved that when $n = 2$ Goddard's conjecture is also true under the additional hypothesis of the compactness of the surface (see also [12] for a simultaneous and independent alternative proof of these facts given by Ramanathan when $n = 2$). In [10], Montiel extended this last result to the general case by showing that the only compact spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$ are the totally umbilical round spheres.

More recently, Cheng and Ishikawa [4] have shown that the totally umbilical round spheres are the only compact spacelike hypersurfaces in de Sitter space with constant scalar curvature $S < n(n-1)$. Li [9] and Zheng [13,14] also obtained interesting characterizations of these hypersurfaces under the hypothesis of constant scalar curvature.

The natural generalization of mean and scalar curvatures for a spacelike hypersurface in de Sitter space are the kth mean curvatures H_k for $k = 1, \dots, n$. Actually, H_1 is the mean curvature and H_2 is, up to a constant, the scalar curvature of the hypersurface. In [2], the first author, jointly with Aledo and Romero, developed some integral formulas for compact spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$ and applied them in order to characterize the totally umbilical round spheres of $\mathbf{S}_1^{n+1}$ as the only compact spacelike hypersurfaces with constant higher order mean curvature, under appropriate hypothesis. In particular, the following theorem was proved in [2].

Theorem 1 (Aledo et al. [2, Theorem 7]). *Let $\psi : M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of $\mathbf{S}_1^{n+1}$. If H_k is constant for some k, $1 \leq k \leq n$, then M^n is a totally umbilical round sphere*.

Since $H_0 = 1$ by definition, the result above can be read as follows: if H_k/H_0 is constant for some k, $1 \leq k \leq n$, then M^n is a totally umbilical round sphere. In this paper, we extend Theorem 1 in the following way.

Theorem 2. *Let $\psi : M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface in de Sitter space which is contained in the chronological future* (*or past*) *of an equator of $\mathbf{S}_1^{n+1}$.*

If H_l does not vanish on M^n and the ratio H_k/H_l is constant for some k, l, $1 \leq l < k \leq n$, then M^n is a totally umbilical round sphere.

Our study is motivated by the recent work of the second author [6–8] (jointly with Lee in [8]), where similar results were established for the case of compact hypersurfaces in Riemannian space forms.

2. Preliminaries

Throughout this paper we will deal with compact spacelike hypersurfaces in de Sitter space. Recall that every compact spacelike hypersurface M^n in $\mathbf{S}_1^{n+1}$ is diffeomorphic to an n-sphere [2] and, in particular, it is orientable. Then, there exists a timelike unit normal field N globally defined on M^n. We will refer to N as the Gauss map of the immersion and we will say that M^n is oriented by N.

We will denote by $A : \chi(M) \to \chi(M)$ the shape operator of M^n in $\mathbf{S}_1^{n+1}$ with respect to N, which is given by

$$A(X) = -\mathrm{d}N(X).$$

Associated to the shape operator of M^n there are n algebraic invariants, which are the elementary symmetric functions σ_k of its principal curvatures $k_1, \ldots, k_n$, given by

$$\sigma_k(k_1, \ldots, k_n) = \sum_{i_1 < \cdots < i_k} k_{i_1} \cdots k_{i_k}, \quad 1 \leq k \leq n.$$

The kth mean curvature H_k of the spacelike hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k \sigma_k(k_1, \ldots, k_n) = \sigma_k(-k_1, \ldots, -k_n).$$

When $k = 1$, $H_1 = -(1/n)\,\mathrm{tr}(A) = H$ is the mean curvature of M^n. On the other hand, when $k = n$, $H_n = (-1)^n \det(A)$ defines the Gauss–Kronecker curvature of the spacelike hypersurface, and for $k = 2$, H_2 is, up to a constant, the scalar curvature S of M^n, since $S = n(n-1)(1 - H_2)$. We refer the reader to [2] for the details.

The proof of our theorem makes an essential use of the following integral formulas for compact spacelike hypersurfaces in $\mathbf{S}_1^{n+1}$, which were developed in [2].

Lemma 3 (Aledo et al. [2, Theorem 2], Minkowski formulas). *Let $\psi : M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ be a compact spacelike hypersurface immersed into de Sitter space and let $a \in \mathbf{L}^{n+2}$ a fixed arbitrary vector. For each $r = 0, \ldots, n-1$ the following formula holds*:

$$\int_M (-H_r\langle a, \psi\rangle + H_{r+1}\langle a, N\rangle)\,\mathrm{d}V = 0, \tag{1}$$

where $\mathrm{d}V$ is the n-dimensional volume element of M^n with respect to the induced metric and the chosen orientation.

3. Proof of the theorem

Let us assume, for instance, that the hypersurface $\psi : M^n \to \mathbf{S}_1^{n+1} \subset \mathbf{L}^{n+2}$ is contained in the future of the equator determined by a unit timelike vector $a \in \mathbf{L}^{n+2}$ (the case of the past is similar). That means that

$$\psi(M) \subset \{x \in \mathbf{S}_1^{n+1} : \langle a, x\rangle < 0\}.$$

Let us orient M^n by the Gauss map N which is in the same time-orientation as a, so that $\langle a, N\rangle \leq -1 < 0$. Since the height function $\langle a, \psi\rangle$ is negative on M^n, by compactness there exists a point $p_0 \in M$ where it attains its maximum

$$\langle a, \psi(p_0)\rangle = \max_{p\in M}\langle a, \psi(p)\rangle < 0.$$

Therefore, its gradient vanishes at that point, $\nabla\langle a, \psi\rangle(p_0) = 0$, and its Hessian satisfies

$$\nabla^2\langle a, \psi\rangle(p_0)(v, w) = -\langle a, \psi(p_0)\rangle\langle v, w\rangle - \langle a, N(p_0)\rangle\langle A_{p_0}(v), w\rangle \leq 0$$

for all $v, w \in T_{p_0}M$ (for the details see the beginning of the proof of Theorem 7 in [2]). On the other hand, since $\langle a, N\rangle^2 = 1 + \langle a, \psi\rangle^2 + |\nabla\langle a, \psi\rangle|^2$ and $\nabla\langle a, \psi\rangle(p_0) = 0$, then

$$-\langle a, N(p_0)\rangle = \sqrt{1 + \langle a, \psi(p_0)\rangle^2}.$$

Therefore, choosing $\{e_1, \ldots, e_n\}$ a basis of principal directions at the point p_0 we conclude that

$$k_i(p_0) \leq \frac{\langle a, \psi(p_0)\rangle}{\sqrt{1 + \langle a, \psi(p_0)\rangle^2}} < 0$$

for each $i = 1, \ldots, n$. In particular, $H_k(p_0)$ and $H_l(p_0)$ are positive, which implies that the constant $c = H_k/H_l = H_k(p_0)/H_l(p_0)$ is positive, and the mean curvature functions H_k and H_l are positive on M^n (recall that H_l does not vanish on M^n by assumption). Therefore, from the proof of Lemma 1 in [11] (see also [[3], Section 12]) and taking into account the sign convention in our definition of the higher order mean curvatures, it follows that every H_r is positive for $r = 1, \ldots, k$ and

$$\frac{H_1}{H_0} \geq \frac{H_2}{H_1} \geq \cdots \geq \frac{H_k}{H_{k-1}},$$

with equality at any stage only at umbilical points, i.e.,

$$H_l H_{k-1} - H_k H_{l-1} \geq 0, \tag{2}$$

with equality at the umbilical points.

Since $H_k = cH_l$ for a constant c by assumption, from the kth Minkowski formula (for $r = k - 1$ in (1)) we obtain that

$$\int_M H_{k-1}\langle a, \psi\rangle\,\mathrm{d}V = c\int_M H_l\langle a, N\rangle\,\mathrm{d}V. \tag{3}$$

On the other hand, the lth Minkowski formula (for $r = l - 1$ in (1)) implies that

$$\int_M H_{l-1}\langle a, \psi\rangle \, \mathrm{d}V = \int_M H_l \langle a, N\rangle \, \mathrm{d}V,$$

so that (3) becomes

$$\int_M (H_{k-1} - cH_{l-1})\langle a, \psi\rangle \, \mathrm{d}V = 0. \tag{4}$$

We also know from (2) that

$$H_{k-1} - cH_{l-1} = \frac{(H_{k-1} - cH_{l-1})H_l}{H_l} = \frac{H_{k-1}H_l - H_k H_{l-1}}{H_l} \geq 0,$$

with equality at the umbilical points. Therefore, since $\langle a, \psi\rangle < 0$ on M^n, (4) implies that $H_{k-1} - cH_{l-1} \equiv 0$ and the hypersurface must be a totally umbilical round sphere.

Acknowledgements

The first author was partially supported by DGICYT Grant No. PB97-0784-C03-02 and Consejería de Educación y Cultura CARM Grant No. PB/5/FS/97, Programa Séneca (PRIDTYC). The second author was supported by Konkuk University in 2000. This work was written while the first author was visiting the Institute des Hautes Études Scientifiques at Bures-sur-Yvette (France). He would like to thank the institution for their wonderful hospitality.

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